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Periodic Interpolation on Uniform Meshes

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INTRODUCTION

In a recent paper Locher [3] has analyzed the interpolation problem produced by the functions

$$g_j = g(\cdot - 2\pi j/n), \qquad j \in \mathbb{Z},\ n \in \mathbb{N}$$

where g is a continuous real valued function on $\mathbb{R}$ having period 2π. He showed that the interpolation problem generated by the function g and its translates g_j has a unique solution if and only if the discrete Fourier coefficients of g,

$$c_{k,n}(g) = \frac{1}{n} \sum_{j=0}^{n-1} g(t_j) \exp(-ikt_j), \qquad t_j = \frac{2\pi j}{n}, \qquad 0 \leq k < n,$$

do not vanish. If g is an odd periodic function, then $c_{0,n}(g) = 0$. Therefore the method of interpolation by translation developed by Locher is not applicable to odd periodic functions g.

The present paper continues the investigations of [3] by weakening the condition $c_{0,n}(g) \neq 0$. We will develop a modification of the method of interpolation by translation. As a first application we obtain the Golomb construction of periodic polynomial splines of odd degree on uniform meshes [2]. Moreover we will use our method to derive explicit constructions of periodic polynomial splines of even degree on uniform meshes and of periodic polynomial midpoint splines on uniform meshes.

1. INTERPOLATION BY TRANSLATION

For the reader's convenience we briefly recall the method of interpolation by translation. With the aid of the 2π-periodic real valued continuous function g we define the functions

$$B_k(t) = \sum_{j=0}^{n-1} g(t - t_j) \exp(ikt_j), \qquad t \in \mathbb{R},\ k \in \mathbb{Z}.$$

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Then we get for any $r \in \mathbb{Z}$

$$B_k(t_r) = \sum_{j=0}^{n-1} g(t_r - t_j) \exp(ikt_j)$$

$$= \sum_{j=r-n+1}^{r} g(t_j) \exp(ikt_{r-j})$$

$$= \exp(ikt_r) \sum_{j=0}^{n-1} g(t_j) \exp(-ikt_j),$$

i.e.,

$$B_k(t_r) = \exp(ikt_r)\, B_k(0), \qquad B_k(0) = nc_{k,n}(g), \qquad k, r \in \mathbb{Z}. \tag{1.1}$$

For $B_k(0) \neq 0$ we introduce the *exponential interpolants*

$$b_k(t) = B_k(t)/B_k(0), \qquad t \in \mathbb{R}, \tag{1,2}$$

which satisfy the interpolation conditions

$$b_k(t_r) = \exp(ikt_r), \qquad r \in \mathbb{Z}. \tag{1.3}$$

It follows from the definition of b_k that

$$b_k \in \operatorname{lin}\{g_0, g_1, ..., g_{n-1}\} =: V_n. \tag{1.4}$$

V_n is the linear space of interpolating functions for the method of interpolation by translation. V_n is translation invariant with respect to the translation $t_1 = 2\pi/n$, i.e., we have

$$f(\cdot - t_1) \in V_n, \qquad f \in V_n. \tag{1.5}$$

THEOREM 1 [3]. *Assume* $B_k(0) \neq 0$ *for* $k = 0, 1, ..., n-1$. *Then*

$$L(t) = \frac{1}{n} \sum_{j=0}^{n-1} b_j(t), \qquad t \in \mathbb{R}, \tag{1.6}$$

is the unique Lagrange function in V_n *satisfying*

$$L(t_k) = \delta_{0,k}, \qquad k = 0, ..., n-1. \tag{1.7}$$

Moreover, for any 2π*-periodic continuous function* f *there is a unique function* $S_n(f) \in V_n$ *that interpolates* f *at the mesh points* t_j, $j \in \mathbb{Z}$.

Proof. It follows from the definition of b_j that $L \in V_n$. Taking into account (1.3) we obtain for $k = 0, ..., n-1$,

$$L(t_k) = \frac{1}{n} \sum_{j=0}^{n-1} b_j(t_k) = \frac{1}{n} \sum_{j=0}^{n-1} \exp(ijt_k) = \delta_{0,k}$$

whence (1.7) follows. The translation invariance of V_n implies that

$$S_n(f) = \sum_{k=0}^{n-1} f(t_k)\, L(\cdot - t_k) \tag{1.8}$$

is the unique function in V_n satisfying

$$S_n(f)(t_k) = f(t_k), \qquad k \in \mathbb{Z}. \tag{1.9}$$

This completes the proof of Theorem 1.

In the sequel we consider some earlier developments on interpolation by translation. Let the generating function g be the sum of an absolutely convergent trigonometric series:

$$g(t) = \sum_{m=-\infty}^{\infty} c_m(g) \exp(imt), \qquad t \in \mathbb{R}. \tag{1.10}$$

Prager [5] assumed

$$c_m(g) > 0, \qquad \sum_{m=-\infty}^{\infty} c_m(g) < \infty, \tag{1.11}$$

and he established Theorem 1 for this special case. As is well known

$$c_{k,n}(g) = \sum_{r=-\infty}^{\infty} c_{k+rn}(g). \tag{1.12}$$

Thus, the relations (1.11) and (1.12) imply

$$B_k(0) \neq 0, \qquad k = 0, ..., n-1 \tag{1.13}$$

in view of (1.1) whence Theorem 1 is applicable. The conditions (1.11) were generalized by Locher [3].

Spline interpolation is associated with the Bernoulli functions P_q, $q \in \mathbb{N}$, which are given by

$$P_q(t) = \sum_{m \neq 0} (im)^{-q} \exp(imt) = \sum_{m=1}^{\infty} \frac{2}{m^q} \cos(mt - \pi q/2). \tag{1.14}$$

It is well known that $P_q(t)$ is a polynomial function of exact degree q for $0 < t < 2\pi$ which may be computed recursively:

$$P_1(t) = \pi - t, \qquad P'_{q+1}(t) = P_q(t), \qquad \int_0^{2\pi} P_{q+1}(t)\, dt = 0,$$

$$0 < t < 2\pi, \qquad q \in \mathbb{N}. \tag{1.15}$$

Using (1.15) we can generate all higher polynomials:

$$\begin{aligned} P_2(t) &= -(\pi-t)^2/2! + \pi^2/6, \\ P_3(t) &= (\pi-t)^3/3! - (\pi-t)\,\pi^2/6, \\ P_4(t) &= -(\pi-t)^4/4! + (\pi-t)^2\,(\pi^2/6)/2! - \pi^4 7/360. \end{aligned} \tag{1.16}$$

It follows from (1.14) and (1.12) that

$$c_{k,n}(P_{2r}) = B_k(0)/n \neq 0, \qquad k = 0,..., n-1,\ r \in \mathbb{N}. \tag{1.17}$$

The interpolating functions generated by P_{2r} are the Bernoulli monosplines introduced in [1].

It follows from (1.6), (1.2), and (1.1) that the periodic Lagrange function L has the alternative representation

$$L(t) = \sum_{k=0}^{n-1} e_k\, g(t-t_k), \qquad e_k = \frac{1}{n}\sum_{j=0}^{n-1} \exp(ijt_k)/B_k(0) \tag{1.18}$$

(see also [3]). Since g is assumed to be real valued we obtain for $n = 2s+1$ the real representation of e_k,

$$\begin{aligned} B'_j &= g(0) + \sum_{r=1}^{s} (g(t_r) + g(t_{n-r}))\cos(jt_r), \\ B''_j &= \sum_{r=1}^{s} (-g(t_r) + g(t_{n-r}))\sin(jt_r), \\ e_k &= \frac{1}{n}\left(1/B'_0 + \sum_{j=1}^{s} 2(\cos(jt_k)\,B'_j + \sin(jt_k)\,B''_j)/(B'^2_j + B''^2_j)\right). \end{aligned} \tag{1.19}$$

2. An Extension of Golomb's Method

It is the main objective of this paper to derive a modified method of interpolation by translation which is applicable to functions $g \in C_{2\pi}$ which

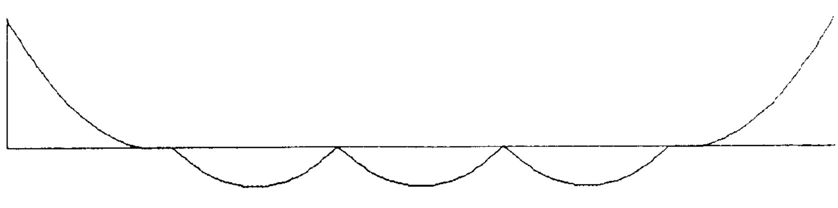

FIG. 1. The Lagrange function for quadratic Bernoulli splines with $n = 5$.

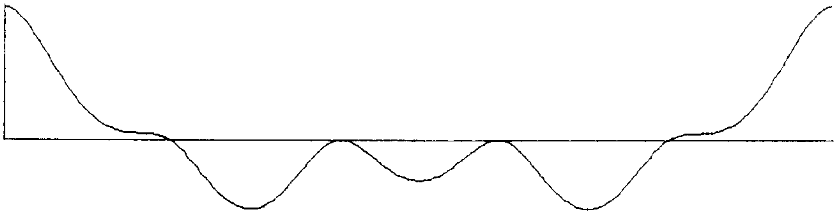

FIG. 2. The Lagrange function for quartic Bernoulli splines with $n = 5$.

merely satisfy $B_k(0) \neq 0$, $0 < k < n$. The basic idea of our method is to replace the function space

$$V_n = \operatorname{lin}\{g(\cdot - t_0), g(\cdot - t_1), \ldots, g(\cdot - t_{n-1})\}$$

by the "first order difference" space

$$V_n^1 = \operatorname{lin}\{1, g(\cdot - t_1) - g, \ldots, g(\cdot - t_{n-1}) - g\}. \tag{2.1}$$

THEOREM 2. *Assume that the 2π-periodic continuous real valued function g satisfies $B_k(0) \neq 0$, $0 < k < n$. Then*

$$l(t) = \frac{1}{n}\left(1 + \sum_{k=1}^{n-1} b_k(t)\right), \qquad t \in \mathbb{R}, \tag{2.2}$$

is the unique periodic Lagrange function in V_n^1 satisfying

$$l(t_k) = \delta_{0,k}, \qquad 0 \leqq k < n. \tag{2.3}$$

Moreover, for any $f \in C_{2\pi}$ there is unique function $Q_n(f) \in V_n^1$ that interpolates f at the mesh points t_l, $j \in \mathbb{Z}$.

Proof. It follows from the definition of the functions b_j, $j = 1, \ldots, n-1$, that l has the representation

$$\begin{aligned} l &= \frac{1}{n}\left(1 + \sum_{j=0}^{n-1} d_j\, g(\cdot - t_j)\right), \\ d_j &= \sum_{k=1}^{n-1} \exp(ikt_j)/B_k(0), \qquad B_k(0) = \sum_{r=0}^{n-1} g(t_r)\exp(-ikt_r). \end{aligned} \tag{2.4}$$

FIG. 3. The Lagrange function for analytic Poission splines associated with $g(t) = (1 - p^2)/(1 + p^2 - 2p\cos(t))$ $(p = 1/2,\ n = 5)$ (see [3]).

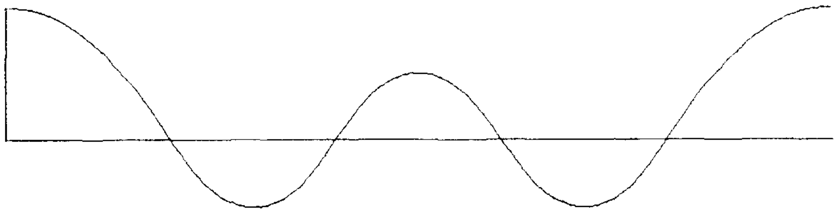

FIG. 4. The Lagrange spline of Sp(2.5).

It is readily established that

$$\sum_{j=0}^{n-1} d_j = 0. \tag{2.5}$$

Thus, we obtain

$$l = \frac{1}{n}\left(1 + \sum_{j=1}^{n-1} d_j(g(\cdot - t_j) - g)\right) \in V_n^1. \tag{2.6}$$

To prove (2.3) we proceed as in the proof of Theorem 1:

$$l(t_k) = \frac{1}{n}\left(1 + \sum_{j=1}^{n-1} b_j(t_k)\right) = \frac{1}{n}\sum_{j=0}^{n-1} \exp(ijt_k) = \delta_{0,k}, \qquad 0 \leqq k < n.$$

Thus, l is a periodic Lagrange function with respect to the mesh points t_j, $j \in \mathbb{Z}$.

To complete the proof of Theorem 1 we first note that V_n^1 is translation invariant with respect to the translation $t_1 = 2\pi/n$, i.e., we have

$$h(\cdot - t_1) \in V_n^1, \qquad h \in V_n^1 \tag{2.7}$$

which follows from

$$g(\cdot - t_j - t_1) - g(\cdot - t_1) = (g(\cdot - t_{j+1}) - g) - (g(\cdot - t_1) - g), \qquad j \in \mathbb{Z}. \tag{2.8}$$

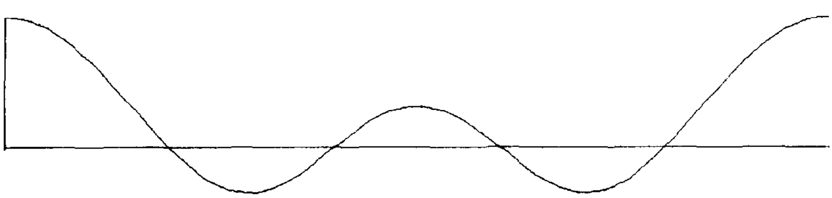

FIG. 5. The Lagrange spline of Sp(4,5).

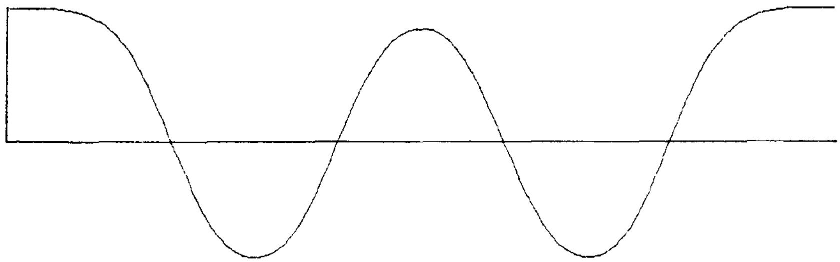

FIG. 6. The Lagrange function for conjugate Poisson splines associated with $g(t) = p\sin(t)/(1+p^2-2p\cos(t))$ for $p=1/2$ and $n=5$.

Thus, we obtain

$$Q_n(f) = \sum_{k=0}^{n-1} f(t_k)\, l(\cdot - t_k) \tag{2.9}$$

which completes the proof of Theorem 2.

Remark. As in Section 1 we have for $n = 2s+1$,

$$\begin{aligned} B_j' &= g(0) + \sum_{r=1}^{s} (g(t_r) + g(t_{n-r}))\cos(jt_r), \\ B_j'' &= \sum_{r=1}^{s} (-g(t_r) + g(t_{n-r}))\sin(jt_r), \\ d_k &= 2\sum_{j=1}^{s} (\cos(jt_k)\, B_j' + \sin(jt_k)\, B_j'')/(B_j'^2 + B_j''^2) \end{aligned} \tag{2.10}$$

since g is assumed to be real valued.

We will use Theorem 2 to prove interpolation theorems for periodic polynomial splines on uniform meshes. Assume $g = P_q$ with $q \geqq 2$. It was proved by Meinardus [4] that the functions

$$1,\ P_q(\cdot - t_1) - P_q, \ldots, P_q(\cdot - t_{n-1}) - P_q$$

FIG. 7. The Lagrange midpoint spline of Sp*(2,5).

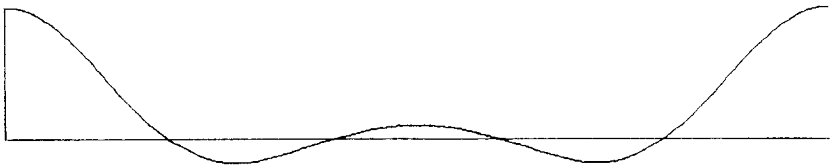

FIG. 8. The Lagrange midpoint spline of Sp*(4,5).

form a basis of the space $\mathrm{Sp}(q-1, n)$ of periodic polynomial splines of degree $q-1$ with spline knots t_j, $j \in \mathbb{Z}$, i.e., we have

$$\mathrm{Sp}(q-1, n) = V_n^1 =: V_n^1(P_q). \tag{2.11}$$

As a first application of Theorem 2 we get Golomb's construction of periodic polynomial splines of degree $2r-1$.

COROLLARY 2.1 [2]. *Suppose that $g = P_{2r}$. Then the function $Q_n(f)$ defined by (2.9) is the unique periodic polynomial spline function of degree $2r-1$ with spline knots t_j, $j \in \mathbb{Z}$, that interpolates f at the points t_j, $j \in \mathbb{Z}$.*

Proof. For $k = 1, \ldots, n-1$ we get

$$\begin{aligned} B_k(0) &= \sum_{u=0}^{n-1} \sum_{m \neq 0} (im)^{-2r} \exp(imt_u - ikt_u) \\ &= \sum_{m \neq 0} (im)^{-2r} \sum_{u=0}^{n-1} \exp(it_u(m-k)) \\ &= \sum_{j=-\infty}^{\infty} (k+nj)^{-2r} (-1)^r n, \end{aligned}$$

i.e., we have $B_k(0) \neq 0$, $k = 1, \ldots, n-1$. Thus, an application of Theorem 2 completes the proof of Corollary 2.1.

COROLLARY 2.2. *Assume $g = P_{2r+1}$ and $n = 2s+1$. Then $Q_n(f)$ is the unique periodic polynomial spline of degree $2r$ with knots t_j, $j \in \mathbb{Z}$, that interpolates f at the points t_j, $j \in \mathbb{Z}$.*

FIG. 9. The Lagrange function for midpoint conjugate Poisson splines with $p = \frac{1}{2}$ and $n = 5$.

Proof. For $k=1,\dots,n-1$ we have

$$\begin{aligned}
B_k(0) &= \sum_{u=0}^{n-1} \sum_{m\neq 0} (im)^{-(2r+1)} \exp(imt_u - ikt_u) \\
&= \sum_{j=-\infty}^{\infty} (k+nj)^{-(2r+1)} (-1)^r n/i \\
&= -i(-1)^r n \sum_{j=0}^{\infty} ((k+nj)^{-(2r+1)} - ((n-k)+nj)^{-(2r+1)}),
\end{aligned}$$

i.e., we have $B_k(0)\neq 0$ for $0<k<n$, $2k\neq n$. Since P_{2r+1} is odd it follows that $B_0(0)=0$ and Theorem 1 is not applicable.

Let $z_j = t_j + \pi/n$, $j\in\mathbb{Z}$. It follows from the basis theorem of Meinardus [4] that the functions

$$1,\ P_q(\cdot - z_1) - P_q, \dots, P_q(\cdot - z_{n-1}) - P_q$$

form a basis of the space $\mathrm{Sp}^*(q-1,n)$ of periodic polynomial midpoint splines of degree $q-1$ with spline knots z_j, $j\in\mathbb{Z}$. Thus, we have for $g = P_q(\cdot - \pi/n)$

$$\mathrm{Sp}^*(q-1,n) = V_n^1 =: V_n^1(P_q(\cdot - \pi/n)). \tag{2.12}$$

COROLLARY 2.3. *Suppose $g = P_{2r+1}(\cdot - \pi/n)$. Then the function $Q_n(f)$ given by* (2.9) *is the unique periodic polynomial midpoint spline of degree* $2r$ *with knots z_j, $j\in\mathbb{Z}$, that interpolates f at the points t_j, $j\in\mathbb{Z}$.*

Proof. Taking into account (1.14) we obtain for $k=1,\dots,n-1$,

$$\begin{aligned}
B_k(0) &= \sum_{m\neq 0} (im)^{-(2r+1)} \sum_{u=0}^{n-1} \exp(im(t_u - \pi/n) - ikt_u) \\
&= \sum_{m\neq 0} \left((im)^{-(2r+1)} \exp(-i\pi m/n) \sum_{u=0}^{n-1} \exp(it_u(m-k)) \right) \\
&= -i(-1)^r n \exp(-i\pi k/n) \sum_{j=-\infty}^{\infty} (-1)^j (k+nj)^{-(2r+1)} \\
&= -i(-1)^r n \exp(-i\pi k/n) \left(\sum_{j=0}^{\infty} (-1)^j (k+nj)^{-(2r+1)} \right. \\
&\qquad\qquad \left. + \sum_{j=1}^{\infty} (-1)^j (k-nj)^{-(2r+1)} \right) \\
&= -i(-1)^r n \exp(-i\pi k/n) \sum_{j=0}^{\infty} (-1)^j ((k+nj)^{-2r-1} \\
&\qquad + (n-k+nj)^{-2r-1}),
\end{aligned}$$

i.e., we have $B_k(0) \neq 0$ for $k = 1, \ldots, n-1$ in view of

$$|B_k(0)| > k^{-2r-1} + (n-k)^{-2r-1} - (k+n)^{-2r-1} - (n-k+n)^{-2r-1}.$$

Again an application of Theorem 2 completes the proof of Corollary 2.3. Since $P_{2r+1}(-t) = -P_{2r+1}(t)$, $z_{n-1-j} = 2\pi - z_j$, it follows that

$$B_0(0) = -\sum_{j=0}^{n-1} P_{2r+1}(z_j) = 0,$$

i.e., Theorem 1 is not applicable to the function $P_{2r+1}(\cdot - \pi/n)$.

Remark. For the functions $g = P_q(\cdot - a)$, $a = 0, \pi/n$, Locher's method works with the space V_n of translates of g which is a space of periodic polynomial splines of degree q with knots of multiplicity 2. In the extension of Golomb's method the set V_n^1 of interpolants is a linear space of periodic polynomial splines of degree $q-1$ with knots of multiplicity 1. For this purpose the local degree of g is diminished by one and put together to $q-1$ by taking the differences from the translates, while the order of differentiability remains $q-2$ so that polynomial degree and this order fit together by their own—furnished by Golomb by the special side condition [2] and (2.11) only.

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